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# On a class of non-self-adjoint periodic eigenproblems with boundary and interior singularities ☆

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ABSTRACT

We prove that all the eigenvalues of a certain highly non-self-adjoint Sturm–Liouville differential operator are real. The results presented are motivated by and extend those recently found by various authors (Benilov et al. (2003) [3], Davies (2007) [7] and Weir (2008) [18]) on the stability of a model describing small oscillations of a thin layer of fluid inside a rotating cylinder.



## 1. The general problem class

We consider on the interval $(-\pi, \pi)$ the singular non-symmetric differential equation

$$Lu := i\varepsilon \frac{d}{dx}\left(f(x)\frac{du}{dx}\right) + i\frac{du}{dx} = \lambda u, \tag{1.1}$$

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in which $f$ is a $2\pi$-periodic function having the following properties:

$$f(x+\pi) = -f(x), \qquad f(-x) = -f(x), \tag{1.2}$$

and also

$$f(x) > 0 \quad \text{for } x \in (0, \pi). \tag{1.3}$$

In particular it follows that $f(\pi\mathbb{Z}) = 0$. We assume that $f$ is continuous, and differentiable except possibly at a finite number of points, the points of non-differentiability excluding $\pi\mathbb{Z}$. We assume that $f'(0) = 2/\pi$ and that $0 < \varepsilon < \pi$.

Our interest in (1.1) is primarily motivated by [3] and [7], where $f(x) = (2/\pi)\sin x$, and therefore (1.1) takes the form

$$i\tilde{\varepsilon}\frac{d}{dx}\left(\sin x \frac{du}{dx}\right) + i\frac{du}{dx} = \lambda u, \tag{1.4}$$

with $0 < \tilde{\varepsilon} < 2$. Eq. (1.4) arises in fluid dynamics, and describes small oscillations of a thin layer of fluid inside a rotating cylinder. It has been long conjectured that, despite the fact that (1.4) is highly non-self-adjoint, all the eigenvalues of (1.4) are real. For a fixed non-zero $\tilde{\varepsilon}$ the conjecture has been supported by some numerical evidence, see e.g. [5], who have also proved that eigenvalues accumulate at infinity along the real line. Davies [7] established that the spectrum is discrete, nonetheless the eigenfunctions do not form a basis. He also obtained a number of very interesting estimates. Recently Weir [17,18] proved, using Davies' estimates, that the spectrum of the highly non-self-adjoint problem (1.4) is indeed purely real and just very recently Davies and Weir [8] studied the asymptotics of the eigenvalues as $\tilde{\varepsilon} \to 0$.

The goal of our paper is to extend Weir's result and to prove that all the eigenvalues of (1.1) are real under minimal restrictions on the coefficient $f(x)$. In order not to hide the fundamental simplicity of our methods under a burden of technical details, the proof that there is no essential spectrum – in fact, the resolvent operator has bounded kernel – will be given in a separate article, [4]. One fundamental condition is the *anti-symmetry* of $f$ with respect to $x$, see (1.2), and also some symmetry properties of the whole operator, see below. This makes (1.1) reminiscent (but not identical) to a wide class of so-called PT-symmetric equations: non-self-adjoint problems, which are not similar to self-adjoint, but which nevertheless possess purely real spectra due to some obvious and hidden symmetries. For examples, surveys, and interesting recent developments see e.g. [1,9,10,15,16,13,14] and references therein.

An additional difficulty in dealing with (1.1) (compared to standard PT-symmetric problems) is the presence of singularities at $x = -\pi, 0, \pi$. On the other hand, it is exactly these singularities which allow us to use a strategy of, very roughly speaking, matching asymptotic expansions and exact solutions.

In the remainder of this section we shall describe some basic properties of the equation and its solutions. Section 2 discusses an illustrative explicit example in which $f$ is piecewise linear, while Section 3 contains the general results. Appendices A and B cover details of some basic asymptotics and spectral theoretical properties of (1.1).

By an elementary Frobenius analysis (see Appendix A, for a rigorous summary of the main asymptotic results) the differential equation (1.1) is seen to possess, for each $\lambda \in \mathbb{C}$, a unique (up to scalar multiples) solution in $L^2(-\pi, \pi)$, which we denote by $\phi(x, \lambda)$. This solution is continuous at $x = 0$, where it is non-vanishing, and may therefore be normalised by the condition

$$\phi(0, \lambda) = 1. \tag{1.5}$$

Any solution linearly independent of $\phi$ will blow up like $x^{-\pi/(2\varepsilon)}$ as $x \to 0$. Later we shall assume the more restrictive condition $0 < \varepsilon < \pi/2$, which is necessary to ensure that the differential equation has a unique solution in a weighted $L^2$-space naturally associated with the problem.

Eq. (1.1) also has singular points at $\pm\pi$; however all solutions are square integrable at these points (in the unweighted space). In the weighted space it will turn out that whether these solutions are square integrable or not depends on how small $\varepsilon$ is.

By the *periodic eigenvalue problem associated with* (1.1) we mean the problem of finding solutions which are square integrable and which satisfy the periodicity condition $u(-\pi) = u(\pi)$. In view of the foregoing comments about the solution $\phi$ it follows that $\mu$ is an eigenvalue if and only if

$$\phi(-\pi, \mu) = \phi(\pi, \mu). \tag{1.6}$$

Now consider, for any $\lambda \in \mathbb{C}$, the function $\psi(x, \lambda) = \phi(-x, \lambda)$. It is easy to see that $\psi$ solves (1.1) with $\lambda$ replaced by $-\lambda$; also, $\psi$ is square integrable and $\psi(0, \lambda) = 1$, hence $\psi(x, \lambda) = \phi(x, -\lambda)$. In other words,

$$\phi(-x, \lambda) = \phi(x, -\lambda). \tag{1.7}$$

In a similar way, taking complex conjugates in (1.1) shows that

$$\overline{\phi(x, \lambda)} = \phi(x, -\bar{\lambda}) = \phi(-x, \bar{\lambda}). \tag{1.8}$$

Eq. (1.8) gives a reflection principle for $\phi$ across the imaginary $\lambda$-axis, on which $\phi$ is real valued.

Using the eigenvalue condition (1.6) and the symmetries (1.7) and (1.8), makes it clear that eigenvalues come in quadruples: $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$. Also, the condition (1.6) for $\mu$ to be an eigenvalue can be written in the equivalent form

$$\phi(\pi, \mu) = \phi(\pi, -\mu) = \overline{\phi(\pi, \overline{\mu})}. \tag{1.9}$$

We would like to show that all eigenvalues of the problem are real.

Finally in this introduction, we emphasise that all of our results can be placed in a proper operator theoretic framework: in particular, the eigenvalues to which we have casually referred are indeed eigenvalues of a closed non-self-adjoint operator; see Appendix B. Let $C^2_{\mathrm{per}}(-\pi, \pi)$ be the space of periodic $C^2$ functions in $(-\pi, \pi)$. The differential expression $L$ with domain $C^2_{\mathrm{per}}(-\pi, \pi)$ defines a closable linear operator acting on the space $L^2(-\pi, \pi)$, despite the fact that $1/f(x)$ is not integrable at $x = 0, \pm\pi$. In order to see this we will show that $L$ is a closed operator in the maximal domain

$$\mathcal{D}_{\mathrm{m}} := \{u \in L^2(-\pi, \pi): Lu \in L^2(-\pi, \pi)\}.$$

Clearly $C^2_{\mathrm{per}}(-\pi, \pi) \subset \mathcal{D}_{\mathrm{m}}$. In Appendix B we will show that $\mathcal{D}_{\mathrm{m}} \subset H^1(-\pi, \pi)$. This ensures that any eigenfunction of $\overline{(L, C^2_{\mathrm{per}}(\pi, \pi))}$ is a solution of the periodic eigenvalue problem associated with (1.1), and vice versa. Note however that in the general case (1.1) we do not claim to prove that the spectrum is purely discrete, for a proof of this statement see [4].

## 2. A piecewise linear example

As an example we start with the case where $f$ is given by

$$f(x) = \begin{cases} -2\pi^{-1}(x + \pi), & -\pi \leqslant x \leqslant -\pi/2, \\ 2\pi^{-1}x, & -\pi/2 \leqslant x \leqslant \pi/2, \\ 2\pi^{-1}(\pi - x), & \pi/2 \leqslant x \leqslant \pi. \end{cases} \tag{2.1}$$

In this case the equation can be solved explicitly in terms of Bessel functions. For $x \in [-\pi/2, \pi/2]$ we have

$$\phi(x,\lambda) = \Gamma(\nu+1)(i\nu\lambda x)^{-\nu/2} J_\nu(2\sqrt{i\nu\lambda x}) =: \zeta_0(x,\lambda), \tag{2.2}$$

where $\nu = \pi/(2\varepsilon)$. For $x \in [\pi/2, \pi]$ we can write

$$\phi(x,\lambda) = A(\lambda)\zeta_1(x-\pi,\lambda) + B(\lambda)\zeta_2(x-\pi,\lambda) \tag{2.3}$$

where

$$\zeta_1(z,\lambda) = (i\nu\lambda z)^{\nu/2} J_\nu(2\sqrt{-i\nu\lambda z}), \tag{2.4}$$

$$\zeta_2(z,\lambda) = (i\nu\lambda z)^{\nu/2} J_{-\nu}(2\sqrt{-i\nu\lambda z}), \tag{2.5}$$

and where $A$ and $B$ must be determined to ensure that $\phi$ is continuous and differentiable at $\pi/2$:

$$\begin{pmatrix} \zeta_1(-\pi/2,\lambda) & \zeta_2(-\pi/2,\lambda) \\ \zeta_1'(-\pi/2,\lambda) & \zeta_2'(-\pi/2,\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) \\ B(\lambda) \end{pmatrix} = \begin{pmatrix} \zeta_0(\pi/2,\lambda) \\ \zeta_0'(\pi/2,\lambda) \end{pmatrix}. \tag{2.6}$$

In view of the fact that $\zeta_1(0,\lambda) = 0$ the condition (1.9) becomes, upon using (2.3),

$$B(\mu)\zeta_2(0,\mu) = B(-\mu)\zeta_2(0,-\mu).$$

The properties of the Bessel functions show that $\zeta_2(0,\lambda) = i^\nu/\Gamma(1-\nu)$ for all $\lambda$, and so the eigenvalue condition becomes

$$B(\mu) = B(-\mu).$$

The solution of (2.6) is given by

$$B(\lambda) = \frac{\det\begin{pmatrix} \zeta_1(-\pi/2,\lambda) & \zeta_0(\pi/2,\lambda) \\ \zeta_1'(-\pi/2,\lambda) & \zeta_0'(\pi/2,\lambda) \end{pmatrix}}{\det\begin{pmatrix} \zeta_1(-\pi/2,\lambda) & \zeta_2(-\pi/2,\lambda) \\ \zeta_1'(-\pi/2,\lambda) & \zeta_2'(-\pi/2,\lambda) \end{pmatrix}}. \tag{2.7}$$

Further progress at this stage appears to rest on explicit calculation with Bessel functions. Firstly, it may be shown (either by reference to Watson's book on Bessel functions or by explicitly solving the first order ODE satisfied by the Wronskian) that

$$\det\begin{pmatrix} \zeta_1(z,\lambda) & \zeta_2(z,\lambda) \\ \zeta_1'(z,\lambda) & \zeta_2'(z,\lambda) \end{pmatrix} = \frac{\sin(\nu\pi)}{\pi}(i\nu\lambda)^\nu z^{\nu-1}. \tag{2.8}$$

Secondly, observe that if we define

$$\chi(x,\lambda) = J_\nu(2\sqrt{i\nu\lambda x}),$$

then

$$\zeta_0(x,\lambda) = \Gamma(\nu+1)(i\nu\lambda x)^{-\nu/2}\chi(x,\lambda) = c\lambda^{-\nu/2}x^{-\nu/2}\chi(x,\lambda),$$

$$\zeta_1(x,\lambda) = (i\nu\lambda x)^{\nu/2}\chi(-x,\lambda) = C\lambda^{\nu/2}x^{\nu/2}\chi(-x,\lambda).$$

An explicit calculation shows that

$$\zeta_0'(\pi/2,\lambda) = c\lambda^{-\nu/2}(\pi/2)^{-\nu/2}\left(\chi'(\pi/2,\lambda) - \frac{\nu}{\pi}\chi(\pi/2,\lambda)\right),$$
$$\zeta_1'(-\pi/2,\lambda) = C\lambda^{\nu/2}(-\pi/2)^{\nu/2}\left(-\chi'(\pi/2,\lambda) - \frac{\nu}{\pi}\chi(\pi/2,\lambda)\right),$$

and hence

$$\det\begin{pmatrix} \zeta_1(-\pi/2,\lambda) & \zeta_0(\pi/2,\lambda) \\ \zeta_1'(-\pi/2,\lambda) & \zeta_0'(\pi/2,\lambda) \end{pmatrix} = 2cC(-1)^{\nu/2}\chi(\pi/2,\lambda)\chi'(\pi/2,\lambda). \tag{2.9}$$

Since $\chi'(x,\lambda) = \sqrt{i\nu\lambda/x}\,J_\nu'(2\sqrt{i\nu\lambda x})$, it follows that for some constant $\tilde{C}$ independent of $\lambda$,

$$B(\lambda) = \tilde{C}\lambda^{1/2-\nu}J_\nu'(\sqrt{2i\nu\lambda\pi}\,)J_\nu(\sqrt{2i\nu\lambda\pi}\,).$$

Hence the eigenvalue condition becomes

$$F(\lambda) = F(-\lambda), \quad \text{where } F(\lambda) = \lambda^{1/2-\nu}J_\nu'(\sqrt{2i\nu\lambda\pi})J_\nu(\sqrt{2i\nu\lambda\pi}). \tag{2.10}$$

Setting $z = \sqrt{2i\nu\lambda\pi}$, we observe that $\sqrt{-2i\nu\lambda\pi} = \pm iz$.

Define

$$\rho(z) := \frac{J_\nu'(z)J_\nu(z)}{J_\nu'(iz)J_\nu(iz)}. \tag{2.11}$$

Then a necessary condition for Eq. (2.10) to be satisfied becomes

$$\left|\rho(z)\right| = 1.$$

**Theorem 2.1.** *For any non-real $\lambda$ (corresponding to $\arg(z) \notin \{\pm\pi/4, \pm3\pi/4\}$), the equation $|\rho(z)| = 1$ cannot be satisfied. In fact, for $-\pi/4 < \arg(\pm z) < \pi/4$, we have $|\rho(z)| < 1$; for $\pi/4 < \arg(\pm z) < 3\pi/4$, we have $|\rho(z)| > 1$. Consequently all the eigenvalues of (1.1), with $f(x)$ given by (2.1), are real.*

**Proof.** For real $\nu$, the zeros of the Bessel function $J_\nu$ and of its derivative are all real. In the sector $-\pi/4 < \arg(z) < \pi/4$, if $z \neq 0$ then $iz$ is not real and so $J_\nu'(iz)J_\nu(iz)$ is non-zero. Consequently $\rho(z)$ is well defined and analytic. Standard asymptotics show that for large $|z|$ in this sector, $|\rho(z)| < 1$. On the rays bounding the sector, $|\rho(z)| = 1$. By the Phragmen–Lindelöf principle, since $\rho$ is a non-constant function which is analytic in a sector of angle $\pi/2$, and has a growth order strictly less than two (in fact, growth order 1), $\rho$ attains its maximum modulus strictly on the boundary rays. Consequently in the sector $-\pi/4 < \arg(z) < \pi/4$, one has $|\rho(z)| < 1$.

The results for the other sectors follow similarly; for instance, if $-3\pi/4 < \arg(z) < -\pi/4$ then $z = -it$ with $-\pi/4 < \arg(t) < \pi/4$, and so $|\rho(z)| = 1/|\rho(t)|$. $\square$

## 3. Results for the general case

For the general case we return to our equation (1.9). Throughout this section we assume $0 < \varepsilon < \pi/2$.

We start with an auxiliary result concerning the location of the zeros of $\phi(\pi, \lambda)$.

**Theorem 3.1.** *All the zeros of $\phi(\pi, \lambda)$ lie on the negative imaginary axis.*

**Proof.** Consider the differential equation (1.1) satisfied by $\phi(x, \lambda)$, restricting attention to the sub-interval $(0, \pi)$. Note that $f$ does not change sign in this sub-interval. Using standard integrating factor techniques we can transform (1.1) into

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = -\frac{i\lambda}{\varepsilon}\frac{p}{f}u, \tag{3.1}$$

where

$$p'/p = f'/f + 1/(\varepsilon f). \tag{3.2}$$

Simple asymptotic analysis shows that

$$p(x) \sim \begin{cases} x^{\pi/(2\varepsilon)} f(x) & (x \text{ near zero}), \\ (\pi - x)^{-\pi/(2\varepsilon)} f(x) & (x \text{ near } \pi). \end{cases} \tag{3.3}$$

Setting $w(x) = p(x)/f(x)$ and $\ell = i\lambda/\varepsilon$, we can write (3.1) in Sturm–Liouville form as

$$-\big(p(x)u'\big)' = \ell w(x)u, \quad x \in (0, \pi). \tag{3.4}$$

Observe that the weight function $w$, which is positive, has very different behaviours at different endpoints:

$$w(x) \sim \begin{cases} x^{\pi/(2\varepsilon)} & (x \text{ near zero}), \\ (\pi - x)^{-\pi/(2\varepsilon)} & (x \text{ near } \pi). \end{cases}$$

By Frobenius analysis, there exist solutions of (1.1) with behaviours $u(x) \sim x^{-\pi/(2\varepsilon)}$ and $u(x) \sim 1$ near $x = 0$, and behaviours $u(x) \sim 1$ and $u(x) \sim (\pi - x)^{\pi/(2\varepsilon)}$ near $x = \pi$. It is therefore easy to check that for $0 < \varepsilon < \pi/2$ there is always precisely one solution in $L^2_w$ near the origin, one solution in $L^2_w$ near $x = \pi$. The problem (3.4) is therefore of limit-point type at the two ends of the interval $(0, \pi)$.

Since $p > 0$ and $w > 0$ on $(0, \pi)$, (3.4) is automatically self-adjoint in $L^2_w$ without the need for boundary conditions. The eigenvalues $\ell$, which must be real, will be precisely those values of $\ell$ for which $\phi(x, \lambda)$, which is always in $L^2_w$ near the origin, is in $L^2_w$ near $x = \pi$. This only happens when $\phi(x, \lambda) \sim (\pi - x)^{\pi/(2\varepsilon)}$, that is, when $\phi(\pi, \lambda) = 0$. Since $\ell = i\lambda/\varepsilon$, this establishes that the zeros of $\phi(\pi, \lambda)$ lie on the imaginary axis in the $\lambda$-plane. Since the eigenvalues of (3.4) are all strictly positive, we can further say that the zeros of $\phi(\pi, \lambda)$ all lie on the negative imaginary $\lambda$-axis. □

**Remark 3.2.** This result, which we have proved for $0 < \varepsilon < \pi/2$, is also true for $0 < \varepsilon < \pi$. In the case $\pi/2 \leqslant \varepsilon < \pi$ the problem is limit circle in $L^2_w$ at $x = \pi$ but the condition $\phi(\pi, \lambda) = 0$ arises naturally by imposing $u(\pi) = 0$ in (3.4), which is the Friedrichs boundary condition for this case.

We now make the transformation $\lambda = iz^2$ and define

$$g(z) = \phi\big(\pi, -iz^2\big). \tag{3.5}$$

The eigenvalue condition $\phi(\pi,\lambda) = \phi(\pi,-\lambda)$ thus becomes

$$g(z) = g(iz). \tag{3.6}$$

In the $\lambda$-plane, by Theorem 3.1, all the zeros of $\phi(\pi,\lambda)$ are of the form $\lambda = -ir$, $r > 0$; hence in the $z$-plane all the zeros of $g$ lie on the imaginary $z$-axis and occur in pairs $\pm i\alpha_n$, $n \in \mathbb{N}$, $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$ (observe that $g(0) \neq 0$). When $z$ is not one of these zeros the function

$$\rho(z) = \frac{g(iz)}{g(z)} \tag{3.7}$$

is well defined and analytic. The eigenvalue condition (3.6) becomes

$$\rho(z) = 1. \tag{3.8}$$

**Theorem 3.3.** *Suppose that the solution $\phi(x,\lambda)$ is, for each $x \in [0,\pi]$, an analytic function of $\lambda$ with growth order at most $1/2$. Then for $-\pi/4 < \arg(\pm z) < \pi/4$, $|\rho(z)| < 1$; for $\pi/4 < \arg(\pm z) < 3\pi/4$, $|\rho(z)| > 1$. Consequently, eigenvalues of (1.1) can only lie on the lines $\arg(z) \equiv \pi/4 \pmod{\pi/2}$, which, in terms of $\lambda = iz^2$, means that all eigenvalues are real.*

**Proof.** We start by outlining the idea of the proof; after the outline, we fill in a technical detail.

Firstly we know that $g(0) \neq 0$, that $g$ is an analytic function of $z^2$ and that the zeros of $g$ occur in pairs $\pm i\alpha_n$. If these zeros have the appropriate asymptotic behaviour then we can write

$$g(z) = \exp\bigl(h\bigl(z^2\bigr)\bigr)\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\alpha_n^2}\right) \tag{3.9}$$

where $h$ is an analytic function and the infinite product is convergent. Since we also know that $g$ has exponential growth order at most 1, it follows that $h$ must be constant, and hence

$$\rho(z) = \frac{g(iz)}{g(z)} = \prod_{n=1}^{\infty}\left(\frac{1 - \frac{z^2}{\alpha_n^2}}{1 + \frac{z^2}{\alpha_n^2}}\right).$$

It is then a simple matter to check that for each $n \in \mathbb{N}$,

$$\left|\frac{1 - \frac{z^2}{\alpha_n^2}}{1 + \frac{z^2}{\alpha_n^2}}\right| < 1 \quad \forall z \text{ such that } -\pi/4 < \arg(z) < \pi/4,$$

and the result is immediate.

It only remains to check that the zeros of $g$ grow sufficiently rapidly to ensure convergence of the infinite product in (3.9). To this end we recall from the proof of Theorem 3.1 that the zeros of $\phi(\pi,\lambda)$ are given by the eigenvalues of the ODE (3.1) on $(0,\pi)$ with Friedrichs boundary conditions. We therefore wish to examine the large eigenvalues of this problem. To this end we make the transformation

$$\phi(x,\lambda) = \frac{1}{\sqrt{p(x)}}\psi(x,\lambda), \tag{3.10}$$

where $p$ is given by (3.2). This reduces the differential equation to

$$-\psi'' + Q(x)\psi = \mu \frac{1}{f}\psi, \tag{3.11}$$

where $Q = p^{-1/2}(p^{-1/2}p')'/2$ and where $\mu = i\varepsilon^{-1}\lambda$ is already known to be real and positive. A simple calculation based on the known asymptotics of $p$ near $x = 0$ and $x = \pi$ shows that

$$Q(x) \sim \begin{cases} \frac{1}{4}(\frac{\pi}{2\varepsilon}+1)(\frac{\pi}{2\varepsilon}-1)x^{-2}, & x \searrow 0, \\ \frac{1}{4}(\frac{\pi}{2\varepsilon}+1)(\frac{\pi}{2\varepsilon}-1)(\pi - x)^{-2}, & x \nearrow \pi. \end{cases}$$

Since $0 < \varepsilon < \pi/2$, we have $Q(x) \to +\infty$ for $x \searrow 0$ and $x \nearrow \pi$. By Sturm Comparison we can therefore obtain lower bounds on the eigenvalues $\mu$ by discarding $Q$ and considering the eigenvalues $\mu$ of the problem

$$-\psi'' - \tilde{Q}\psi = \mu \frac{1}{f(x)}\psi, \quad x \in (0, \pi),$$

where $\tilde{Q} > 0$ is a sufficiently large constant. Standard WKB estimates (see, e.g., Bender and Orszag [2]) give for the $n$th eigenvalue the formula

$$\int_0^\pi \sqrt{\frac{\mu_n}{f(x)}}\,dx \sim n,$$

where we observe that the integral converges because $f$ has simple zeros at the endpoints of the interval. Hence

$$\mu_n = O\big(n^2\big).$$

Recalling the transformations $\lambda = iz^2$ and $\lambda = i\varepsilon^{-1}\mu$ and the definition of the zeros $\alpha_n$ of $g$ and their correspondence with the eigenvalues of this problem, we must have

$$\lambda_n = i\alpha_n^2 = i\varepsilon^{-1}\mu_n,$$

whence

$$\alpha_n = O(n).$$

This is sufficient to ensure convergence of the product in (3.9). $\square$

**Remark 3.4.** In the case of the piecewise linear coefficient example of Eq. (2.1), $\phi(x, \lambda)$ has growth order 1/2 in $\lambda$ and hence $\phi(x, -iz^2)$ has growth order 1 in $z$, for any $x \in [0, \pi]$. This follows from the explicit expressions for the solutions given in terms of Bessel functions in Section 2.

The same conclusion also holds for the case $f(x) = (2/\pi)\sin(x)$ considered by Davies [7]: the equation admits a change of variables $t = \tan(x/2)$, and the solutions are then given in this case in terms of Heun G-functions on the interval $[0, +\infty)$. For relevant asymptotic results, see e.g. [12].

In both of these cases, therefore, the eigenvalues of the corresponding eigenvalue problem are all real.

The following lemma widens the class of functions $f$ for which the function $\phi(x, \lambda)$ has growth order 1/2 in $\lambda$ for all $x \in [0, \pi]$.

**Lemma 3.5.** *Suppose that $f$ falls within the class introduced in Section* 1 *and that $f$ is linear in arbitrarily small neighbourhoods of $x = 0$, $x = \pi$. Then $\phi(x, \lambda)$ has growth order $1/2$ in $\lambda$ for all $x \in [0, \pi]$.*

**Proof.** Suppose that $f(x) = 2x/\pi$ for $x \in [0, \delta]$ and that $f(x) = 2(\pi - x)/\pi$ for $x \in [\pi - \delta, \pi]$ for some $\delta > 0$. Then the explicit expression for $\phi$ given in (2.2) is valid for all $x \in [0, \delta]$ and, with standard asymptotics of Bessel functions, establishes that $\phi(x, \lambda)$ and $\phi'(x, \lambda)$ have exponential growth order $1/2$ in $\lambda$ for all such $x$. Writing the differential equation for $\phi$ as

$$\frac{d}{dx}\begin{pmatrix} \phi(x,\lambda) \\ \lambda^{-1/2} f(x)\phi'(x,\lambda) \end{pmatrix} = \begin{pmatrix} 0 & \lambda^{1/2}/f(x) \\ -i\lambda^{1/2}/\varepsilon & -1/(\varepsilon f) \end{pmatrix}\begin{pmatrix} \phi(x,\lambda) \\ \lambda^{-1/2} f(x)\phi'(x,\lambda) \end{pmatrix} \tag{3.12}$$

we observe that this is a system of the form $Y' = A(x, \lambda)Y$ in which $\|A(x, \lambda)\| \leqslant C_\delta|\lambda|^{1/2}$ for all $x \in [\delta, \pi - \delta]$. By standard Picard estimates

$$\left\|\begin{pmatrix} \phi(x,\lambda) \\ \lambda^{-1/2} f(x)\phi'(x,\lambda) \end{pmatrix}\right\| \leqslant \left\|\begin{pmatrix} \phi(\delta,\lambda) \\ \lambda^{-1/2} f(\delta)\phi'(\delta,\lambda) \end{pmatrix}\right\| \exp\bigl(C_\delta(x-\delta)|\lambda|^{1/2}\bigr)$$

for all $x \in [\delta, \pi - \delta]$. This establishes that the result of our lemma holds for all $x \in [0, \pi - \delta]$. Finally we must extend the result over $[\pi - \delta, \pi]$. To this end we write

$$\phi(x,\lambda) = A(\lambda)\zeta_1(x-\pi,\lambda) + B(\lambda)\zeta_2(x-\pi,\lambda), \quad x \in [\pi-\delta, \pi], \tag{3.13}$$

where $\zeta_1$ and $\zeta_2$ are the functions introduced in Eqs. (2.4), (2.5), both of which have growth order $1/2$ in $\lambda$ for all $x$. The coefficients may be found by solving the system

$$\begin{pmatrix} \zeta_1(-\delta,\lambda) & \zeta_2(-\delta,\lambda) \\ p(\delta)\zeta_1'(-\delta,\lambda) & p(\delta)\zeta_2'(-\delta,\lambda) \end{pmatrix}\begin{pmatrix} A(\lambda) \\ B(\lambda) \end{pmatrix} = \begin{pmatrix} \phi(\delta,\lambda) \\ p(\delta)\phi'(\delta,\lambda) \end{pmatrix} \tag{3.14}$$

where $p$ may be any function which is non-zero at $\delta$ but is most conveniently chosen according to (3.2). This ensures that the determinant of the matrix on the left-hand side of (3.14) is independent of $\delta$ and may be calculated from its value as $\delta \to 0$. From the asymptotics of $\zeta_1$ and $\zeta_2$ it turns out that this value is a non-zero constant, and so

$$\begin{pmatrix} A(\lambda) \\ B(\lambda) \end{pmatrix} = c(\lambda)\begin{pmatrix} p(\delta)\zeta_2'(-\delta,\lambda) & -\zeta_2(-\delta,\lambda) \\ -p(\delta)\zeta_1'(-\delta,\lambda) & \zeta_1(-\delta,\lambda) \end{pmatrix}\begin{pmatrix} \phi(\delta,\lambda) \\ p(\delta)\phi'(\delta,\lambda) \end{pmatrix}.$$

Every quantity appearing on the right-hand side has growth order at most $1/2$ in $\lambda$ and so $A(\lambda)$ and $B(\lambda)$ have growth order $1/2$ in $\lambda$. Thus from (3.13) it follows that $\phi(x, \lambda)$ has growth order $1/2$ for all $x \in [\pi - \delta, \pi]$. □

**Remark 3.6.** The ideal way to obtain our final result now would be to show that $\phi(x, \lambda)$ always has growth order $1/2$ for the whole class of coefficients $f$ introduced in Section 1. Unfortunately this does not appear to be easy, since growth orders are not necessarily preserved in approximation limits. For instance, $\exp(\lambda^2)$ has growth order 2 but can be approximated locally uniformly by Taylor polynomials, which all have growth order 0. Nevertheless the following result is true.

**Theorem 3.7.** *Let $f$ be a function of the class introduced in Section* 1 *which additionally has the property that $f''(0)$ and $f''(\pi)$ exist. Then all the eigenvalues of eigenvalue problem* (1.1) *are real.*

**Proof.** By Theorem 3.3 and Lemma 3.5 the result holds for functions $f$ which are linear in arbitrarily small neighbourhoods of the endpoints.

Now suppose that we have a problem for which $f$ is not linear near the endpoints. Nevertheless we may approximate $f$ by a function which is linear on $[0,\delta]$ and $[\pi-\delta,\pi]$ for small $\delta$. From (1.6) and (1.7), we know that the condition for $\mu$ to be an eigenvalue is that $\mu$ be a zero of the function

$$d(\lambda)=\phi(\pi,\lambda)-\phi(\pi,-\lambda).$$

We shall attach subscripts $\delta$ to the quantities where $f$ is replaced by a linear function on $[0,\delta]\cup[\pi-\delta,\pi]$: thus we shall approximate $\phi(x,\lambda)$ by $\phi_\delta(x,\lambda)$ and $d(\lambda)$ by $d_\delta(\lambda)$. If we can show that

$$\lim_{\delta\searrow 0}\big(d_\delta(\lambda)-d(\lambda)\big)=0 \tag{3.15}$$

locally uniformly in $\lambda$, then the zeros of $d(\lambda)$ will be precisely the limits of the zeros of $d_\delta(\lambda)$. This is achieved, for instance, by using the argument principle to count zeros inside any contour, exploiting the fact that for analytic functions the locally uniform convergence of the functions implies locally uniform convergence of their derivatives. Since the zeros of $d_\delta$ all lie on the real axis, we shall have proved our result.

Choose $f_\delta$ as follows. On $[0,\delta]\cup[\pi-\delta,\pi]$ $f_\delta$ should be linear and it should also match the values of $f$ at 0, $\delta$, $\pi-\delta$ and $\pi$: consequently we shall have

$$f_\delta(0)=f(0)=0,\qquad f_\delta(\delta)=f(\delta),\qquad f_\delta(\pi-\delta)=f(\pi-\delta),\qquad f_\delta(\pi)=f(\pi)=0,$$

also, as $\delta\searrow 0$,

$$f_\delta'(0)\to f'(0),\qquad f_\delta'(\pi)\to f'(\pi).$$

Frobenius analysis (more precisely, the uniform asymptotics in Lemma A.4) shows that

$$\phi(x,\lambda)=1-\frac{i\lambda}{1+\varepsilon f'(0)}x+o(x),\qquad \phi'(x,\lambda)=-\frac{i\lambda}{1+\varepsilon f'(0)}+o(1) \tag{3.16}$$

for small $x$. It is convenient also to have the second solution $\Phi(x,\lambda)$ which satisfies

$$\Phi(x,\lambda)=x^{-1/(\varepsilon f'(0))}\big(1+O(x)\big),\qquad \Phi'(x,\lambda)=\frac{-1}{\varepsilon f'(0)}x^{-1-1/(\varepsilon f'(0))}\big(1+O(x)\big) \tag{3.17}$$

for small $x$.

Note that we also have similar asymptotics for $\phi_\delta$ and $\Phi_\delta$:

$$\phi_\delta(x,\lambda)=1-\frac{i\lambda}{1+\varepsilon f_\delta'(0)}x+o(x),\qquad \phi_\delta'(x,\lambda)=-\frac{i\lambda}{1+\varepsilon f_\delta'(0)}+o(1),$$

$$\Phi(x,\lambda)=x^{-1/(\varepsilon f_\delta'(0))}\big(1+O(x)\big),\qquad \Phi'(x,\lambda)=-\frac{1}{\varepsilon f_\delta'(0)}x^{-1-1/(\varepsilon f_\delta'(0))}\big(1+O(x)\big).$$

Moreover the correction terms can be bounded locally uniformly in $x\in[0,\delta]$ by the remarks following Lemma A.4. Consequently, since $f_\delta'(0)-f'(0)=o(1)$,

$$\phi(\delta,\lambda)-\phi_\delta(\delta,\lambda)=o(\delta),\qquad \phi'(\delta,\lambda)-\phi_\delta'(\delta,\lambda)=o(1).$$

All of these bounds are locally uniform in $\lambda$.

Now on $[\delta, \pi - \delta]$ we have

$$\phi_\delta(x, \lambda) = c_1 \phi(x, \lambda) + c_2 \Phi(x, \lambda) \tag{3.18}$$

where the coefficients $c_1$ and $c_2$ are chosen to ensure continuity and differentiability at $x = \delta$. In terms of the coefficient $p$ introduced in (3.2), and suppressing the $\lambda$-dependence for simpler notation, we have

$$\begin{pmatrix} \phi(\delta) & \Phi(\delta) \\ p\phi'(\delta) & p\Phi'(\delta) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \phi_\delta(\delta) \\ p\phi_\delta'(\delta) \end{pmatrix},$$

whence, in terms of the determinant $\Delta$ of the matrix on the left-hand side of this system,

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{p(\delta)}{\Delta} \begin{pmatrix} \Phi'(\delta)(\phi_\delta(\delta) - \phi(\delta)) - (\phi_\delta'(\delta) - \phi'(\delta))\Phi(\delta) \\ \phi(\delta)(\phi_\delta'(\delta) - \phi'(\delta)) - \phi'(\delta)(\phi_\delta(\delta) - \phi(\delta)) \end{pmatrix}.$$

Bearing in mind that from (3.3),

$$p(\delta) = O\bigl(\delta^{1+1/(\varepsilon f'(0))}\bigr)$$

and that $\Delta$ is a $\delta$-independent constant, we obtain by elementary estimates that

$$c_1 = 1 + o(\delta), \qquad c_2 = o\bigl(\delta^{1+1/(\varepsilon f'(0))}\bigr).$$

From (3.18) it follows that for each fixed $x \in [0, \pi - \delta]$ we have

$$\lim_{\delta \searrow 0} \phi_\delta(x, \lambda) = \phi(x, \lambda)$$

locally uniformly in $\lambda$. We need to extend this convergence to $x = \pi$.

To this end we introduce the solutions $\psi(x)$ and $\Psi(x)$ for the unperturbed system and $\psi_\delta(x)$ and $\Psi_\delta(x)$ for the perturbed system, determined by the asymptotic behaviours (see Lemma A.4 again)

$$\begin{aligned} \psi(x) &= 1 + \frac{i\lambda}{1 - \varepsilon f'(\pi)}(\pi - x) + o(\pi - x), \\ \Psi(x) &= (\pi - x)^{1/(\varepsilon f'(\pi))}\bigl(1 + O(\pi - x)\bigr), \end{aligned} \tag{3.19}$$

$$\begin{aligned} \psi_\delta(x) &= 1 + \frac{i\lambda}{1 - \varepsilon f_\delta'(0)}(\pi - x) + o(\pi - x), \\ \Psi_\delta(x) &= (\pi - x)^{1/(\varepsilon f_\delta'(\pi))}\bigl(1 + O(\pi - x)\bigr). \end{aligned} \tag{3.20}$$

We write

$$\phi_\delta(x) = c_{1,\delta}\psi_\delta(x) + c_{2,\delta}\Psi_\delta(x)$$

and

$$\phi(x) = c_1\psi(x) + c_2\Psi(x).$$

In view of the asymptotics it follows that

$$\phi(\pi) = c_1, \qquad \phi_\delta(\pi) = c_{1,\delta}.$$

Therefore in order to establish that $\lim_{\delta\searrow 0}\phi_\delta(\pi,\lambda)=\phi(\pi,\lambda)$ locally uniformly in $\lambda$ it is sufficient for us to prove that $c_{1,\delta}$ converges locally uniformly to $c_1$. Now

$$\begin{pmatrix} \psi_\delta(\pi-\delta) & \Psi_\delta(\pi-\delta) \\ p\psi_\delta'(\pi-\delta) & p\Psi_\delta'(\pi-\delta) \end{pmatrix} \begin{pmatrix} c_{1,\delta} \\ c_{2,\delta} \end{pmatrix} = \begin{pmatrix} \phi_\delta(\pi-\delta) \\ p\phi_\delta'(\pi-\delta) \end{pmatrix},$$

and a similar equation holds with $\delta=0$. A calculation similar to the ones performed before shows that

$$c_{1,\delta}=c_1+\frac{p}{\Delta}\left\{\left(\Psi_\delta'-\Psi'\right)\phi_\delta+\Psi'(\phi_\delta-\phi)-(\Psi_\delta-\Psi)\phi_\delta'-\Psi\left(\phi_\delta'-\phi'\right)\right\}\Big|_{x=\pi-\delta}.$$

We now estimate the various terms, bearing in mind that from (3.3) we have

$$p(\pi-\delta)=O\left(\delta^{1-1/(\varepsilon f'(\pi))}\right)$$

which may be large. Consider, for example, the term

$$p(\pi-\delta)\left(\Psi_\delta'(\pi-\delta)-\Psi'(\pi-\delta)\right)\phi_\delta(\pi-\delta).$$

In this term we have

$$\Psi_\delta'(\pi-\delta)-\Psi'(\pi-\delta)=O\left(\delta^{1/(\varepsilon f'(\pi))-1}\left(\delta^{1/(\varepsilon f_\delta'(\pi))-1/(\varepsilon f'(\pi))}-1\right)\right).$$

The multiplicative factor of $\delta^{1/(\varepsilon f'(\pi))-1}$ cancels with the $p(\pi-\delta)$ term, while the $\phi_\delta(\pi-\delta)$ term is close to $\phi(\pi-\delta)$ which remains bounded as $\delta\searrow 0$, since all solutions of the unperturbed equation are bounded in a neighbourhood of $\pi$. This leaves a term

$$O\left(\delta^\nu-1\right)$$

in which

$$\nu=\varepsilon^{-1}\frac{f'(\pi)-f_\delta'(\pi)}{f_\delta'(\pi)f'(\pi)}.$$

However the fact that $f$ is twice differentiable at $\pi$ ensures from the construction of $f_\delta$ that

$$f_\delta'(\pi)-f'(\pi)=\frac{\delta}{2}f''(\pi)+o(1)$$

and so the term which is $O(\delta^\nu-1)$ is actually

$$O\left(\delta^{\delta f''(\pi)/2}-1\right).$$

This tends to zero as $\delta$ tends to zero, regardless of whether $f''(\pi)$ be positive or negative.

The other terms may be dealt with in a similar way. Since all the asymptotics are locally uniform in $\lambda$, we get

$$\lim_{\delta\searrow 0}\phi_\delta(\pi,\lambda)=\phi(\pi,\lambda)$$

locally uniformly with respect to $\lambda$. The result follows. $\square$

## Appendix A. Asymptotics

In this appendix we examine the asymptotic behaviour of solutions $u$ of (1.1) and their derivatives, separately near $x = 0$ and $x = \pi$. In order to have a completely rigorous treatment in each case we shall transform the problem to an infinite interval and invoke the Levinson Theorem [11, Theorem 1.3.1].

Consider first the behaviour in a neighbourhood of 0. Make the change of variable $x = \exp(-t)$ so that 0 maps to $\infty$. Eq. (1.1) now becomes

$$i\varepsilon \frac{d}{dt}\left(e^t f\left(e^{-t}\right)\frac{du}{dt}\right) - i\frac{du}{dt} = \lambda e^{-t} u. \tag{A.1}$$

Under our hypotheses on $f$ we know that for all $x$ in a neighbourhood of 0,

$$f(x) = xf'(0) + r(x),$$

where $|r(x)| \leqslant Cx^2$ for some constant $C > 0$. As a consequence

$$e^t f\left(e^{-t}\right) = f'(0) + \rho(t),$$

where

$$\left|\rho(t)\right| \leqslant C\exp(-t). \tag{A.2}$$

Consequently we can write (A.1) as a first order system

$$\frac{d}{dt}\begin{pmatrix} u \\ (f'(0)+\rho(t))u' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{f'(0)+\rho(t)} \\ \frac{-i\lambda}{\varepsilon}e^{-t} & \frac{\varepsilon^{-1}}{f'(0)+\rho(t)} \end{pmatrix}\begin{pmatrix} u \\ (f'(0)+\rho(t))u' \end{pmatrix} \tag{A.3}$$

or as

$$\frac{d\mathbf{u}}{dt} = \left(A_\infty + R(t)\right)\mathbf{u}, \tag{A.4}$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ (f'(0)+\rho(t))u' \end{pmatrix}, \qquad A_\infty = \begin{pmatrix} 0 & 1/f'(0) \\ 0 & \varepsilon^{-1}/f'(0) \end{pmatrix},$$

and

$$\left\|R(t)\right\| \leqslant Ce^{-t}. \tag{A.5}$$

The matrix $A_\infty$ has eigenvalues 0 and $(\varepsilon f'(0))^{-1}$ and is diagonalisable:

$$A_\infty = V\Lambda V^{-1},$$

where $\Lambda = \operatorname{diag}(0, (\varepsilon f'(0))^{-1})$,

$$V = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}.$$

Consequently the system (A.4) can be diagonalised by the transformation $\mathbf{v} = V^{-1}\mathbf{u}$, and

$$\frac{d\mathbf{v}}{dt} = \big(\Lambda + V^{-1}R(t)V\big)\mathbf{v}. \tag{A.6}$$

**Lemma A.1.** *Suppose that $T$ is chosen sufficiently large to ensure that*

$$\int_T^\infty \big\| V^{-1}R(t)V \big\|\, dt < 1/2.$$

*Then the system* (A.6) *has solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ such that*

$$\mathbf{v}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{r}_1(t), \tag{A.7}$$

$$\mathbf{v}_2(t) = \exp\big(t/\big(\varepsilon f'(0)\big)\big)\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbf{r}_2(t)\right\}, \tag{A.8}$$

*where, for all $t \geqslant T$,*

$$\max\big(\big\|\mathbf{r}_1(t)\big\|, \big\|\mathbf{r}_2(t)\big\|\big) \leqslant 2\int_t^\infty \big\| V R(s) V^{-1} \big\|\, ds. \tag{A.9}$$

*Consequently, in view of* (A.5) *and the transformation $\mathbf{v} = V^{-1}\mathbf{u}$, the linear system* (A.4) *possesses solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ such that*

$$\mathbf{u}_1(t) = V\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_1(t), \tag{A.10}$$

$$\mathbf{u}_2(t) = \exp\big(t/\big(\varepsilon f'(0)\big)\big)\left\{V\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \eta_2(t)\right\}, \tag{A.11}$$

*where, for some constant $C > 0$ and for all $t \geqslant T$,*

$$\max\big(\big\|\eta_1(t)\big\|, \big\|\eta_2(t)\big\|\big) \leqslant C\exp(-t). \tag{A.12}$$

**Proof.** This is just the statement of Levinson's Theorem for our problem. □

It is now a simple matter to translate this lemma into information about $u(x)$, $u'(x)$ and $f(x)u'(x)$ for $x$ in a neighbourhood of 0:

**Lemma A.2.** *For Eq.* (1.1), *there exist solutions $u_1$ and $u_2$ such that*

$$\begin{pmatrix} u_1(x) \\ -f(x)u_1'(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_1(x), \tag{A.13}$$

$$\begin{pmatrix} u_2(x) \\ -f(x)u_2'(x) \end{pmatrix} = x^{-1/(\varepsilon f'(0))}\left\{\begin{pmatrix} \varepsilon \\ 1 \end{pmatrix} + \eta_2(x)\right\}, \tag{A.14}$$

*where $\|\eta_1(x)\| \leqslant Cx$, $\|\eta_2(x)\| \leqslant Cx$ for all $0 \leqslant x \leqslant x_0$, where $x_0$ is some positive real number. Moreover the error bounds are locally uniform in $\lambda$ and depend only on $\lambda$, $\varepsilon$ and $\sup_{0<x<x_0} |r(x)|$.*

Observe that the solution $u_1$ is the solution $\phi$ which is regular near zero, used throughout our article.

A similar lemma can be proved concerning behaviour near $x = \pi$:

**Lemma A.3.** *For Eq.* (1.1)*, there exist solutions $u_3$ and $u_4$ such that*

$$\begin{pmatrix} u_3(x) \\ -f(x)u_3'(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta_3(x), \tag{A.15}$$

$$\begin{pmatrix} u_4(x) \\ -f(x)u_4'(x) \end{pmatrix} = (\pi - x)^{1/(\varepsilon f'(0))}\left\{\begin{pmatrix} -\varepsilon \\ 1 \end{pmatrix} + \eta_4(x)\right\}, \tag{A.16}$$

*where $\|\eta_1(x)\| \leqslant C(\pi - x)$, $\|\eta_2(x)\| \leqslant C(\pi - x)$ for all $\pi - x_0 \leqslant x \leqslant \pi$, where $x_0$ is some positive real number. Moreover the error bounds are locally uniform in $\lambda$ and depend only on $\lambda$, $\varepsilon$ and $\sup_{-x_0<x<0} |r(x)|$.*

These results are sufficient for most of our article, but not quite for the estimates in the proof of Theorem 3.7. In particular, although Lemma A.2 gives the results (3.17) and (3.20), it does not quite give the results (3.16) and (3.19), since the terms $\frac{i\lambda}{1+\varepsilon f'(0)}x$ are absorbed into the error terms. The solution to this problem relies on using an integral equation, Eq. (1.4.13) in [11, p. 12], according to which the solution $\mathbf{v}_1$ of Lemma A.1 will satisfy

$$\mathbf{v}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_t^\infty F(t)F^{-1}(s)V^{-1}R(s)V\mathbf{v}_1(s)\,ds, \tag{A.17}$$

in which, for our system,

$$F(t) = \begin{pmatrix} 1 & 0 \\ 0 & \exp(t/(\varepsilon f'(0))) \end{pmatrix}. \tag{A.18}$$

The higher quality approximation is achieved by replacing $\mathbf{v}_1$ in the integral on the right-hand side by its approximation from Lemma A.1, so that

$$V^{-1}R(s)V\mathbf{v}_1(s) = V^{-1}R(s)V\begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\big(\exp(-2s)\big). \tag{A.19}$$

Suppose that $f''(0)$ and $f''(\pi)$ exist. Some explicit calculations show that

$$V^{-1}R(s)V = \mathrm{e}^{-s}\begin{pmatrix} \lambda i & \lambda i\varepsilon \\ -\lambda i\varepsilon^{-1} & -\lambda i - f''(0)/(2\varepsilon f'(0)^2) \end{pmatrix} + \mathrm{e}^{-s}\begin{pmatrix} 0 & 0 \\ 0 & g(s) \end{pmatrix}$$

where

$$g(t) = O\big(\mathrm{e}^t\big(\mathrm{e}^t f\big(\mathrm{e}^{-t}\big) - f'(0)\big) - f''(0)/2\big) \tag{A.20}$$

tends to zero as $t \to \infty$. Hence

$$V^{-1}R(s)V\mathbf{v}_1(s) = \mathrm{e}^{-s}\lambda i\begin{pmatrix} 1 \\ -\varepsilon^{-1} \end{pmatrix} + O\big(\mathrm{e}^{-2s}\big). \tag{A.21}$$

Substituting (A.18) and (A.21) back into (A.17) and transforming back to the original variables yields the following result:

**Lemma A.4.** *The solution $u_1$ of* (1.1) *mentioned in Lemma* A.2 *has an asymptotic form*

$$\begin{pmatrix} u_1(x) \\ -f(x)u_1'(x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\lambda i x}{1+\varepsilon f'(0)} \begin{pmatrix} -1 \\ f'(0) \end{pmatrix} + \eta_5(x), \tag{A.22}$$

*where $\|\eta_5(x)\| \leqslant C(x^2 + |x^{-1}f(x) - f'(0) - xf''(0)/2|)$ for all $x$ in some interval $[0, x_0]$, $x_0 > 0$, and the constant $C$ depends only on $\lambda$, $\varepsilon$ and $\sup_{0<x<x_0} |r(x)|$.*

As an immediate corollary of this lemma, and in particular of the form of the error term, if one changes $f$ in a neighbourhood $[0, \delta]$ to some function $f_\delta$, then the same asymptotics will hold in $[0, \delta]$ with $f_\delta'(0)$ replacing $f'(0)$ on the right-hand side of (A.22). A similar reasoning works near $x = \pi$ and gives a rigorous justification of the asymptotic results used in the proof of Theorem 3.7.

## Appendix B. Linear operator realisation of *L*

The quadratic form associated to $L$ with domain $C^2(-\pi, \pi)$ is strongly indefinite. In fact, its numerical range is the whole of $\mathbb{C}$. In this appendix we show that, in spite of this strong indefiniteness, $L$ is a closable operator on $C^2(-\pi, \pi)$. The key to this result depends on an analysis of the behaviour of $u \in \mathcal{D}_{\mathrm{m}}$ in neighbourhoods of 0, $\pm\pi$.

**Lemma B.1.** *Let $0 < \varepsilon < \pi/2$. There exists a constant $c_\varepsilon > 0$, such that*

$$\int_{-\pi}^{\pi} \left|u'(x)\right|^2 dx \leqslant c_\varepsilon \int_{-\pi}^{\pi} \left|Lu(x)\right|^2 dx \tag{B.1}$$

*for all $u \in \mathcal{D}_{\mathrm{m}}$.*

**Proof.** Suppose that

$$-iLu = \varepsilon\bigl(fu'\bigr)' + u' = v \in L^2(-\pi, \pi). \tag{B.2}$$

We will only show that

$$\int_{0}^{\pi} \left|u'(x)\right|^2 dx \leqslant c_\varepsilon \int_{0}^{\pi} \left|v(x)\right|^2 dx.$$

If we restrict our attention to $x \in [0, \pi]$, Eq. (B.2) can be written as

$$\bigl(pu'\bigr)' = \frac{p}{\varepsilon f} v,$$

where $p$ is given by (3.2) and satisfies the asymptotics (3.3). Note that $p(x)$ as well as $f(x)$ are continuous and positive in $(0, \pi)$. By virtue of Hardy's inequality (B.3) below, we can find a constant $a_\varepsilon > 0$ such that

$$\int_{0}^{\pi/2} \left|u'(x)\right|^2 dx \leqslant a_\varepsilon \int_{0}^{\pi/2} \frac{|p(x)u'(x)|^2}{x^{2+\pi/\varepsilon}} dx \leqslant a_\varepsilon \int_{0}^{\pi/2} \frac{|(p(x)u'(x))'|^2}{x^{\pi/\varepsilon}} dx$$

$$= \frac{a_\varepsilon}{\varepsilon^2} \int_0^{\pi/2} \frac{|p(x)/f(x)|^2 |v(x)|^2}{x^{\pi/\varepsilon}} \, dx \leqslant \frac{a_\varepsilon}{\varepsilon^2} \int_0^{\pi/2} \left|v(x)\right|^2 dx.$$

Similarly, if we change variables $y = \pi - x$, we can find $b_\varepsilon > 0$ such that

$$\int_0^{\pi/2} \left|u'(y)\right|^2 \leqslant b_\varepsilon \int_0^{\pi/2} \frac{|p(y)u'(y)|^2}{y^{2-\pi/\varepsilon}} \, dy \leqslant \frac{b_\varepsilon}{\varepsilon^2} \int_0^{\pi/2} \left|v(y)\right|^2 dy.$$

This ensures the desired inequality. □

In the proof of this lemma observe that Hardy's inequality

$$\frac{(1-a)^2}{4} \int_0^b t^{a-2} \left|w(t)\right|^2 dt \leqslant \int_0^b t^a \left|w'(t)\right|^2 dt \tag{B.3}$$

holds for any $a \in \mathbb{R}$ if $w$ is absolutely continuous and $\operatorname{supp}(w) \subset (0, b)$. See [6, Section 5.3].

The fact that $L$ is closed in the maximal domain $\mathcal{D}_{\mathrm{m}}$ is a straightforward consequence of this lemma.

**Corollary B.2.** *Let $L$ denote the operator defined by taking the closure of the differential expression at the left side of* (1.1) *in $C^2_{\mathrm{per}}(-\pi, \pi)$. If the resolvent set of $L$ is non-empty, then the spectrum of $L$ is the set of eigenvalues of the periodic problem associated with* (1.1)*.*

**Proof.** Suppose that the resolvent set of $L$ is non-empty, and let $\lambda$ be a member of this set. Then, by Lemma B.1, $(L-\lambda)^{-1}$ is a bounded operator mapping $L^2(-\pi, \pi)$ into $H^1(-\pi, \pi)$. As $H^1(-\pi, \pi)$ is compactly contained in $L^2(-\pi, \pi)$, then necessarily $(L-\lambda)^{-1}$ is a compact operator. The spectrum of $L$ therefore consists of isolated eigenvalues of finite multiplicity. These are precisely the set of eigenvalues of the periodic problem associated with (1.1). □

Note that if $f(x) = (2/\pi)\sin(x)$ the hypotheses of this corollary are satisfied. See [7] and [17].

## References


[1] C.M. Bender, S. Boettcher, Real spectra in non-Hermitian Hamiltonians having symmetry, Phys. Rev. Lett. 80 (1998) 5243–5246.
[2] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw–Hill, Singapore, 1978.
[3] E.S. Benilov, S.B.G. O'Brien, I.A. Sazonov, A new type of instability: explosive disturbances in a liquid film inside a rotating horizontal cylinder, J. Fluid Mech. 497 (2003) 201–224.
[4] L. Boulton, M. Levitin, M. Marletta, On a class of nonselfadjoint periodic boundary value problems with discrete real spectrum, in: Amer. Math. Soc. Transl., 2010, in press.
[5] M. Chugunova, D. Pelinovsky, Spectrum of a non-self-adjoint operator associated with the periodic heat equation, J. Math. Anal. Appl. 342 (2008) 970–988.
[6] E.B. Davies, Spectral Theory and Differential Operators, Cambridge University Press, Cambridge, 1995.
[7] E.B. Davies, An indefinite convection–diffusion operator, LMS J. Comput. Math. 10 (2007) 288–306.
[8] E.B. Davies, J. Weir, Convergence of eigenvalues for a highly non-self-adjoint differential operator, Bull. London Math. Soc. 42 (2) (2010) 237–249, doi:10.1112/blms/bdp120.
[9] P. Dorey, C. Dunning, R. Tateo, Spectral equivalences, Bethe ansatz equations, and reality properties in $\mathcal{PT}$-symmetric quantum mechanics, J. Phys. A 34 (28) (2001) 5679–5704.
[10] P. Dorey, C. Dunning, R. Tateo, The ODE/IM correspondence, J. Phys. A 40 (2007) R205–R283.
[11] M.S.P. Eastham, The Asymptotic Solution of Linear Differential Systems, Clarendon Press, Oxford, 1989.
[12] M.V. Fedoryuk, Asymptotics of the spectrum of the Heun equation and of Heun functions, Izv. Akad. Nauk SSSR Ser. Mat. 55 (3) (1991) 631–646 (in Russian); translation in: Math. USSR Izv. 38 (3) (1992) 621–635.

[13] H. Langer, C. Tretter, A Krein space approach to $PT$-symmetry, Czechoslovak J. Phys. 54 (10) (2004) 1113–1120; Czechoslovak J. Phys. 56 (9) (2006) 1063–1064, Corrigendum.
[14] D. Krejčiřík, H. Bíla, M. Znojil, Closed formula for the metric in the Hilbert space of a $\mathcal{PT}$-symmetric model, J. Phys. A 39 (32) (2006) 10143–10153.
[15] K.C. Shin, On the reality of the eigenvalues for a class of $\mathcal{PT}$-symmetric oscillators, Comm. Math. Phys. 229 (3) (2002) 543–564.
[16] K.C. Shin, Eigenvalues of $\mathcal{PT}$-symmetric oscillators with polynomial potentials, J. Phys. A 38 (27) (2005) 6147–6166.
[17] J. Weir, An indefinite convection–diffusion operator with real spectrum, Appl. Math. Lett. 22 (2) (2009) 280–283.
[18] J. Weir, Correspondence of the eigenvalues of a non-self-adjoint operator to those of a self-adjoint operator, preprint, http://arxiv.org/abs/0801.4959v2, 2008.